Fidelity freeze for a random matrix model with off-diagonal perturbation

H.-J. Stöckmann[1,*] and H. Kohler[2,†]
[1]*Fachbereich Physik der Philipps-Universität Marburg, Renthof 5, D-35032 Marburg, Germany*
[2]*Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 19, D-69120 Heidelberg, Germany*
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The concept of fidelity has been introduced to characterize the stability of a quantum-mechanical system against perturbations. The fidelity amplitude is defined as the overlap integral of a wave packet with itself after the development forth and back under the influence of two slightly different Hamiltonians. It was shown by Prosen and Žnidarič in the linear-response approximation that the decay of the fidelity is frozen if the Hamiltonian of the perturbation contains off-diagonal elements only. In the present work the results of Prosen and Žnidarič are extended by a supersymmetry calculation to arbitrary strengths of the perturbation for the case of an unperturbed Hamiltonian taken from the Gaussian orthogonal ensemble and a purely imaginary antisymmetric perturbation. It is found that for the exact calculation the freeze of fidelity is only slightly reduced as compared to the linear-response approximation. This may have important consequences for the design of quantum computers.

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I. INTRODUCTION

The concept of fidelity was originally introduced by Peres [1] to characterize the quantum-mechanical stability of a system against perturbations. Recently it enjoys a renewed popularity because of its obvious relevance for quantum computing. In the present context the works that focus on random matrix aspects are of particular relevance. First the paper by Gorin *et al.* [2] has to be mentioned, where the Gaussian average of the decay of the fidelity amplitude was calculated in a linear response approximation. For small perturbations the authors found a predominantly Gaussian decay, with a crossover to exponential decay for strong perturbations, in accordance with the literature [3,4]. The results of the paper could be experimentally verified in an ultrasound experiment [5] and in a microwave billiard [6,7]. Using supersymmetry techniques, the limitations of the linear response approximation could be overcome, yielding analytic expressions for the decay of the fidelity amplitude for the Gaussian orthogonal (GOE) and Gaussian unitary (GUE) ensemble. Quite surprisingly, a recovery of the fidelity was found at the Heisenberg time, [8] which was interpreted as a spectral analog of a Debye-Waller factor [9]. Reference [8] is the basis for the present work.

The Gaussian decay observed for small perturbation is caused by the diagonal part of the perturbation in the eigenbasis of the unperturbed Hamiltonian. This was the motivation for Prosen and Žnidarič to look for perturbations with a zero diagonal, first in classically integrable systems [10]. Later on they extended their studies to classically chaotic systems [11]. In linear response approximation they found a plateau in the decay of the fidelity. Only after extraordinarily long times the decay started again, exponentially below, and Gaussian beyond the Heisenberg time. It remained, however, an open question of whether this freeze of the fidelity is reality or whether it is just an artifact of the approximation. It was the motivation for the present work to answer this question by extending the previous supersymmetry calculation to the freeze situation. It will be shown that the freeze is present also in the exact calculation. This may have important consequences for quantum computing. If one succeeds in imbedding the atoms representing the qubits into an environment coupled only via an off-diagonal perturbation to the atoms, an enhancement of the system's stability by orders of magnitude is expected.

The present results are not restricted to random matrices. In Ref. [12] it is shown that, e.g., kicked tops with a corresponding dynamics follow exactly the random matrix predictions of the present paper.

II. THE LINEAR RESPONSE APPROXIMATION

The fidelity amplitude is defined as the overlap integral of an initial wave function $|\psi\rangle$ with itself after the time evolution due to two slightly different Hamiltonians H_0 and $H_\lambda = H_0 + \lambda V$,

$$f_\lambda(\tau) = \langle \psi | e^{2\pi \imath H_\lambda \tau} e^{-2\pi \imath H_0 \tau} | \psi \rangle, \tag{1}$$

where the time τ is given in units of the Heisenberg time. Expanding the initial wave packet in terms of eigenfunctions to H_0, $|\psi\rangle = \Sigma a_n |\psi_n\rangle$, Eq. (1) may be written as

$$f_\lambda(\tau) = \sum_{n,m} a_n a_m^* \langle \psi_n | e^{2\pi \imath H_\lambda \tau} e^{-2\pi \imath H_0 \tau} | \psi_m \rangle. \tag{2}$$

Now an average over all initial wave functions is performed. In chaotic systems, in contrast to integrable ones, the a_n are uncorrelated, i. e. $\langle a_n a_m \rangle = (1/N)\delta_{nm}$, where N is the number of eigenfunctions entering the sum (2). One then ends up with

*Electronic address: stoeckmann@physik.uni-marburg.de
†Electronic address: kohler@tphys.uni-heidelberg.de

$$\langle f_\lambda(\tau)\rangle = \frac{1}{N}\langle \mathrm{tr}[e^{2\pi\iota H_\lambda \tau}e^{-2\pi\iota H_0\tau}]\rangle, \tag{3}$$

where the brackets denote an ensemble average. It may be worthwhile to note that up until now all experiments studying the fidelity decay measure ensemble averages. This is, in particular, the case for the ultrasound and microwave experiments mentioned in the Introduction [5–7], but it is true as well for all spin-echo experiments where always ensemble averages over a large number of probe spins are observed.

Under the assumptions that (i) H_0 is taken either from the Gaussian orthogonal (GOE) or the Gaussian unitary ensemble (GUE) with a mean level spacing of one in the band center and (ii) that the variances of the matrix elements of V are given by

$$\langle V_{ij}V_{kl}\rangle = \begin{cases} \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} & (\mathrm{GOE}),\\ \delta_{il}\delta_{jk} & (\mathrm{GUE}).\end{cases} \tag{4}$$

Gorin *et al.* [2] obtained for the fidelity amplitude in the linear response approximation,

$$\langle f_\epsilon(\tau)\rangle \sim 1 - \epsilon C(\tau), \tag{5}$$

where $\epsilon=4\pi^2\lambda^2$, and $C(\tau)$ is given by

$$C(\tau) = \frac{\tau^2}{\beta} + \frac{\tau}{2} - \int_0^\tau \int_0^t b_{2,\beta}(t')dt'\, dt. \tag{6}$$

$b_{2,\beta}(\tau)$ is the two-point form factor, and β is the universality index, i.e., $\beta=1$ for the GOE, and $\beta=2$ for the GUE. For the Gaussian ensembles, $b_{2,\beta}(\tau)$ is known, and $C(\tau)$ can be explicitly calculated [2]. The range of validity of the linear response approximation can be somewhat extended, by exponentiating Eq. (5),

$$\langle f_\epsilon(\tau)\rangle = e^{-\epsilon C(\tau)}. \tag{7}$$

The authors argued that the errors of the approximation should be fairly small for $\lambda\sim 0.1$ and negligible for $\lambda \sim 0.01$ (corresponding to $\epsilon=0.4$ and 0.004, respectively), which was fully confirmed by the exact calculations [8].

The Gaussian decay for small perturbations is caused by the diagonal part of the perturbation. This is immediately evident from Eq. (3): For small perturbations V can be truncated to V_{diag}, its diagonal part in the basis of eigenfunctions of H_0. In this regime, Eq. (3) reduces to

$$\begin{aligned}\langle f_\lambda(\tau)\rangle &= \frac{1}{N}\langle \mathrm{tr}\, e^{2\pi\iota(H_0+\lambda V_{\mathrm{diag}})\tau}e^{-2\pi\iota H_0\tau}\rangle\\ &= \frac{1}{N}\left\langle \sum_n e^{2\pi\iota(E_n+\lambda V_{nn})\tau}e^{-2\pi\iota E_n\tau}\right\rangle = \frac{1}{N}\left\langle \sum_n e^{2\pi\iota\lambda V_{nn}\tau}\right\rangle\\ &= \frac{1}{N}\langle \mathrm{tr}\, e^{2\pi\iota\lambda V_{\mathrm{diag}}\tau}\rangle = e^{-(\epsilon/2)\tau^2\langle V_{\mathrm{diag}}^2\rangle},\end{aligned} \tag{8}$$

where the E_n are the eigenenergies of H_0 [4]. This suggests to consider perturbations with vanishing diagonal matrix elements in the eigenbasis of H_0 [13]. In the linear response approximation one then obtains

$$\langle f_\epsilon(\tau)\rangle = e^{-\epsilon C_{\mathrm{freeze}}(\tau)}, \tag{9}$$

where C_{freeze} differs from the expression (6) derived previously only by the fact that the term τ^2/β is missing on the right hand side of the equation (6). The resulting decay of the fidelity amplitude is extremely slow. It will be discussed later and compared with the exact result, as obtained from the supersymmetry calculation.

III. THE PURELY IMAGINARY ANTISYMMETRIC PERTURBATION

To apply the supersymmetric technique of Ref. [8], the Hamiltonian needs to be invariant under the action of the orthogonal/unitary group. This is not the case for a GOE perturbation with a deleted diagonal. However, a purely imaginary antisymmetric matrix meets with both requirements, zero diagonal elements and orthogonal symmetry. Therefore it is an ideal candidate. We consider the Hamiltonian

$$H_\lambda = H_0 + \iota\lambda V, \tag{10}$$

where H_0 is taken from the GOE, i.e.,

$$\langle (H_0)_{ij}(H_0)_{kl}\rangle_{H_0} = \frac{N}{\pi^2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{11}$$

and V is real antisymmetric, i.e.,

$$\langle V_{ij}V_{kl}\rangle_V = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{12}$$

The variance of the matrix elements has been chosen to have a mean level spacing of one for H_0 and of order $1/\sqrt{N}$ for V. Note that in contrast to Ref. [8] the mean density of states does not remain constant with increasing perturbation, but decreases with increasing λ. In fact, it is irrelevant in the present context, whether a defolding to a constant mean density of states is performed or not. Such a defolding would imply an additional factor of $1/\sqrt{1+(\pi\lambda)^2/N}$ on the right hand side of Eq. (10), which in the final limit $N\to\infty$ reduces to one. These definitions are consistent with the normalization used by Gorin *et al.* [2]. Expressing $\langle f_\epsilon(\tau)\rangle$, see Eq. (3), in terms of its Fourier transform,

$$\langle f_\epsilon(\tau)\rangle = \int dE_1\, dE_2\, e^{2\pi\iota(E_1-E_2)\tau}R_\epsilon(E_1,E_2), \tag{13}$$

we have

$$R_\epsilon(E_1,E_2) \propto \frac{1}{N}\left\langle \mathrm{tr}\left(\frac{1}{E_{1-}-H_0-\iota\lambda V}\frac{1}{E_{2+}-H_0}\right)\right\rangle, \tag{14}$$

with $E_\pm = E\pm\iota\eta$. We rewrite $R_\epsilon(E_1,E_2)$ using the formula

$$\mathrm{tr}\frac{1}{AB} = \frac{1}{2}\left|\sum_{n,m}\frac{\partial}{\partial J_{nm}}\frac{\partial}{\partial K_{mn}}\frac{\det(A+J)}{\det(A-J)}\frac{\det(B+K)}{\det(B-K)}\right|_{J=K=0}. \tag{15}$$

In our case $A=E_{2+}-H_0$ is real symmetric and $B=E_{1-}-H_0-\iota\lambda V$ is Hermitian. According to the universality classes of A and B, we write the determinants as Gaussian integrals over real and complex wave functions, respectively. We obtain

$$\sum_{n,m} \frac{\partial}{\partial J_{nm}} \frac{\det(A+J)}{\det(A-J)}\bigg|_{J=0} = -i \int d[x]d[y]d[\xi]d[\xi^*] \times \sum_{n,m} (x_n x_m + y_n y_m - \xi_n^* \xi_m - \xi_m^* \xi_n) \times e^{-ix^T A x - iy^T A y - i\xi^\dagger A \xi - i\xi^T A \xi^*}, \quad (16)$$

where the commuting integration variables are real. We adopt the usual convention and use Latin letters for commuting, and Greek ones for anticommuting variables, respectively. For B we obtain, instead,

$$\sum_{n,m} \frac{\partial}{\partial K_{mn}} \frac{\det(B+K)}{\det(B-K)}\bigg|_{K=0} = i \int d[a]d[b]d[\eta]d[\eta^*] \times \sum_{n,m} (a_m a_n + b_m b_n - \eta_m^* \eta_n - \eta_n^* \eta_m) \times e^{i(z^\dagger B z + 2\eta^\dagger B \eta)}, \quad (17)$$

where z_i is complex, and a_i and b_i are its real and imaginary part, respectively. Collecting the results, we arrive at

$$R_\epsilon(E_1,E_2) \propto \frac{1}{N} \int d[a]d[b]d[x]d[y]d[\xi]d[\xi^*]d[\eta]d[\eta^*] e^{-\iota\left[E_{2+}\sum_n (x_n^2+y_n^2+2\xi_n^*\xi_n) - E_{1-}\sum(a_n^2+b_n^2+2\eta_n^*\eta_n)\right]} \times \sum_{n,m} (x_n x_m + y_n y_m - \xi_n^* \xi_m - \xi_n \xi_m^*) \times (a_m a_n + b_m b_n - \eta_m^* \eta_n - \eta_m \eta_n^*) \left\langle e^{\iota\lambda \sum_{n,m} V_{nm}(a_n b_m - a_m b_n - \iota \eta_n^* \eta_m + \iota \eta_m^* \eta_n)} \right\rangle_V \times \left\langle e^{-\iota \sum_{n,m} H_{0nm}(x_n x_m + y_n y_m - a_n a_m - b_n b_m + \xi_n^* \xi_m + \xi_m^* \xi_n - \eta_n^* \eta_m - \eta_m^* \eta_n)} \right\rangle_{H_0}. \quad (18)$$

The commuting integration variables are all real. Now the average is taken over real symmetric H_0 using Eq. (11)

$$\langle \cdots \rangle_{H_0} = \exp\left(-\frac{N}{\pi^2} \mathrm{Str}(LZ)^2\right), \quad (19)$$

where Z is a supermatrix given by $Z = \Sigma_n \mathbf{z}_n \mathbf{z}_n^\dagger$, with $\mathbf{z}_n^T = (x_n, y_n, \xi_n, \xi_n^*, a_n, b_n, \eta_n, \eta_n^*)$, and $L = \mathrm{diag}(1_4, -1_4)$ in the advanced-retarded block notation. Z is exactly the matrix given in Table 4.1 of Ref. [14], denoted by VWZ in the following. It has an orthosymplectic symmetry, i.e., in Boson-Fermion block notation the Boson-Boson block is real symmetric and the Fermion-Fermion block is Hermitian self-dual. For the V average we obtain with Eq. (12),

$$\langle \cdots \rangle_V = \exp[-\lambda^2 \,\mathrm{Str}(\mathbf{K}T)^2]. \quad (20)$$

Here $T = \Sigma_n \mathbf{a}_n \mathbf{a}_n^\dagger$ with $\mathbf{a}_n^T = (a_n, b_n, \eta_n, \eta_n^*)$, and Str is the supertrace in the notation of VWZ. The supermatrix $\mathbf{K}$ is given by

$$\mathbf{K} = \begin{pmatrix} -\sigma_y & 0 \\ 0 & \sigma_z \end{pmatrix}, \quad (21)$$

where we used the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (22)$$

In $\mathbf{K}$, the off-diagonal nature of the perturbation is encoded.

The subsequent steps are the same as described in [8,14]. After transforming Eqs. (19) and (20) by means of two Hubbard-Stratonovich transformations, the integrations over the a,b,x,y variables, and over the auxiliary variables of one Hubbard-Stratonovich transformation can be performed, resulting in

$$R_\epsilon(E_1,E_2) \propto \frac{\pi^4}{4N^3} \int d[\sigma] \mathrm{Str}(\mathbf{P}\sigma_{RA}^\dagger \mathbf{P}\sigma_{RA}) \times e^{-(\pi^2/4N)\mathrm{Str}\,\sigma^2} e^{(\pi^4/4N^2)\lambda^2\,\mathrm{Str}(\mathbf{K}\sigma_{RR})^2} \times \left[\mathrm{Sdet}\begin{pmatrix} \sigma_{AA} - E_{1-} & \sigma_{RA}^\dagger \\ \sigma_{RA} & \sigma_{RR} - E_{2+} \end{pmatrix}\right]^{-N/2}. \quad (23)$$

Sdet denotes the superdeterminent, and $\mathbf{P} = \mathrm{diag}(1,1,-1,-1)$. Terms vanishing in the limit $N \to \infty$ have been discarded. The matrix σ has the same orthosymplectic symmetry as Z and reads in advanced-retarded block notation,

$$\sigma = \begin{bmatrix} \sigma_{AA} & \sigma_{RA}^\dagger \\ \sigma_{RA} & \sigma_{RR} \end{bmatrix}. \quad (24)$$

Introducing the notation $E_{1/2} = \bar{E} \pm E/2$, and substituting σ_{AA} and σ_{RR} by $\sigma_{AA} + E/2$ and $\sigma_{RR} - E/2$, respectively, we obtain

$$R_\epsilon(E_1,E_2) \propto \frac{\pi^4}{4N^3} \int d[\sigma] \mathrm{Str}(\mathbf{P}\sigma_{RA}^\dagger \mathbf{P}\sigma_{RA}) \times e^{-(\pi^2/4N)[\sigma^2 + E\,\mathrm{Str}(\sigma_{AA} - \sigma_{RR})]} \times e^{(\pi^4\lambda^2/4N^2)[\mathrm{Str}(\mathbf{K}\sigma_{RR})^2 - E\,\mathrm{Str}\,\mathbf{K}\sigma_{RR}]} \times [\mathrm{Sdet}(\sigma - \bar{E})]^{-N/2}. \quad (25)$$

This expression can be evaluated in the limit $N \to \infty$ by a saddle point approximation.

IV. SADDLE POINT APPROXIMATION

The next steps are a direct repetition of the corresponding ones in Ref. [8]. We shall adopt the notation from VWZ, which is the main source for the following calculations. First we diagonalize σ,

$$\sigma = T_0^{-1} R^{-1} \sigma_D R T_0, \tag{26}$$

where R is block diagonal and

$$T_0 = \begin{pmatrix} \sqrt{1 + t_{12}t_{21}} & \iota t_{12} \\ -\iota t_{21} & \sqrt{1 + t_{21}t_{12}} \end{pmatrix} \tag{27}$$

[see VWZ, Eqs. (5.28) and (5.29)]. The integration of the diagonal variables of σ can be performed by means of the saddle point approximation. σ_D at the saddle point reads as

$$\sigma_D = \begin{pmatrix} s_A & \mathbf{0} \\ \mathbf{0} & s_R \end{pmatrix}, \tag{28}$$

where the advanced and retarded saddle points are given by

$$s_{A/R} = \frac{1}{2}(\bar{E} \pm \iota\Delta),$$

$$\Delta = \frac{2N}{\pi}\sqrt{1 - \left(\frac{\pi\bar{E}}{2N}\right)^2} = \frac{2N}{\pi}\rho, \tag{29}$$

and ρ is the density of states. In the following we shall restrict ourselves to the band center, $\bar{E}=0$, where Eq. (29) reduces to $s_{A/R}=\pm\iota\pi/N$. We then have for the matrix σ at the saddle point,

$$\sigma = \frac{N}{\pi}\begin{pmatrix} \iota(\mathbf{1} + 2t_{12}t_{21}) & 2t_{12}\sqrt{\mathbf{1} + t_{21}t_{12}} \\ 2t_{21}\sqrt{\mathbf{1} + t_{12}t_{21}} & \iota(-\mathbf{1} - 2t_{21}t_{12}) \end{pmatrix}. \tag{30}$$

The matrix R [see Eq. (26)] does not enter, since it commutes with σ_D at the saddle point. We obtain

$$R_\epsilon(E_1,E_2) \propto \frac{\pi^4}{4N^3}\int \mathcal{F}(t_{12}) d[t_{12}] \mathrm{Str}(\mathbf{P}\sigma_{AR}\mathbf{P}\sigma_{RA}) \times e^{-(\pi^2/4N)[E\ \mathrm{Str}(\sigma_{AA}-\sigma_{BB}) - (\pi^2\lambda^2/N)\mathrm{Str}(\mathbf{K}\sigma_{AA})^2]}, \tag{31}$$

where the integral is over the elements of the matrix t_{12}, parametrizing the saddle-point manifold. The function $\mathcal{F}(t_{12})=\mathrm{Sdet}^{-1/2}(1+t_{12}t_{21})$ is the Berezinian of the coordinate transformation, Eq. (27). Using Eq. (30), the various terms entering Eq. (31) may be written as

$$\mathrm{Str}(\sigma_{AA} - \sigma_{RR}) = \iota\frac{4N}{\pi}\mathrm{Str}(t_{12}t_{21}), \tag{32}$$

$$\mathrm{Str}(\mathbf{K}\sigma_{RR})^2 = \left(\frac{\iota N}{\pi}\right)^2 \mathrm{Str}[\mathbf{K}(\mathbf{1} + 2t_{12}t_{21})]^2 \tag{33}$$

$$\mathrm{Str}(\sigma_{AR}\mathbf{P}\sigma_{RA}\mathbf{P}) = \left(\frac{2N}{\pi}\right)^2 \mathrm{Str}(t_{21}\sqrt{\mathbf{1} + t_{12}t_{21}}\mathbf{P} \times t_{12}\sqrt{\mathbf{1} + t_{21}t_{12}}\mathbf{P}). \tag{34}$$

We proceed further by diagonalizing the matrices t_{12} and t_{21}. This is achieved by the radial decomposition,

$$t_{12} = U_1^{-1}MUU_2, \quad t_{21} = U_2^{-1}U^\dagger MU_1, \tag{35}$$

with diagonal $M=\mathrm{diag}(\mu_1,\mu_2,\iota\mu,\iota\mu)$. The matrix $U=1_2 \oplus \hat{U}$ is a 4×4 block diagonal matrix, with $\hat{U}\in \mathrm{SU}(2)$ [see VWZ, Eq. (I.18)]. The U_i $(i=1,2)$ may be parametrized as $U_i=V_iO_i$, where the $O_i=\hat{O}_i\oplus 1_2$ are 4×4 block diagonal, with $\hat{O}_i\in \mathrm{SO}(2)$. The parametrization of the V_i in terms of anticommuting variables is postponed to Appendix A. If we, moreover, introduce

$$X = M^2 = \mathrm{diag}(x,y,-z,-z),$$

$$x = \mu_1^2, \quad y = \mu_2^2, \quad z = \mu^2, \tag{36}$$

we can write Eqs. (32)–(34), using Eq. (35) and Eq. (36)

$$\mathrm{Str}(\sigma_{AA} - \sigma_{RR}) = \iota\frac{4N}{\pi}\mathrm{Str}\, X, \tag{37}$$

$$\mathrm{Str}(\mathbf{K}\sigma_{RR})^2 = \left(\frac{\iota N}{\pi}\right)^2 \mathrm{Str}[\mathbf{K_1}(\mathbf{1} + 2X)]^2, \tag{38}$$

$$\mathrm{Str}(\sigma_{AR}\mathbf{P}\sigma_{RA}\mathbf{P}) = \left(\frac{2N}{\pi}\right)^2 \mathrm{Str}(\sqrt{X}\sqrt{\mathbf{1}+X}\mathbf{P_1} \times \sqrt{X}\sqrt{\mathbf{1}+X}\mathbf{P_2}), \tag{39}$$

where

$$\mathbf{K_1} = U_1\mathbf{K}U_1^{-1}, \quad \mathbf{P_1} = U_1\mathbf{P}U_1^{-1}, \quad \mathbf{P_2} = UU_2\mathbf{P}U_2^{-1}U^{-1}. \tag{40}$$

Under the transformations, Eq. (35) and Eq. (36), the measure transforms as

$$d[t_{12}] = \mathcal{G}(X)d\mu(U_1)d\mu(U_2)d\mu(U)d[X]. \tag{41}$$

The function $\mathcal{G}$ has been calculated in VWZ [Eq. (K.17)]. The average in Eq. (31) is over the elements of the matrices X,U,U_1,U_2. Only $\mathbf{P_2}$ depends on the matrix elements of U_2. It will be shown in Appendix A that $U_2\mathbf{P}U_2^{-1}$ averaged over the matrix elements of U_2 is nothing but a multiple of the four-dimensional unit matrix. Thus the U dependence cancels. We are then left with an average over x,y,z, and the matrix elements of U_1. Inserting these results into Eq. (31), we get

$$R_\epsilon(E_1,E_2) \propto \frac{1}{N}\int \mathcal{F}(X)\mathcal{G}(X)d[X]d\mu(U_1)\mathrm{Str}[X(X+\mathbf{1})\mathbf{P_1}] \times e^{-\pi\iota E\ \mathrm{Str}\, X - (\epsilon/16)\mathrm{Str}[(\mathbf{1}+2X)\mathbf{K_1}]^2}, \tag{42}$$

where we employed the definition $\epsilon=4\pi^2\lambda^2$; previously shown. The Berezinians $\mathcal{F}(X)$ and $\mathcal{G}(X)$ can be comprised in one measure function $\mu(X)$ that was given in VWZ:

$$\mu(X) = \frac{|x-y|}{\sqrt{xy(x+1)(y+1)}} \frac{z(1-z)}{(z+x)^2(z+y)^2}. \tag{43}$$

Substituting expression (42) for $R_\epsilon(E_1,E_2)$ into Eq. (13), and introducing $E=(E_1-E_2)/2$ and $\bar{E}=(E_1+E_1)/2$ as new integration variables, the E integration generates a delta function, whereas the $\bar{E}$ integration corresponds to an energy average. The result is (see Ref. [8] for details)

$$\langle f_\epsilon(\tau)\rangle \propto \frac{1}{N}\int_0^\infty du \int_0^\infty dv \int_0^1 dz\, \mu(u,v,z) d\mu(U_1)\delta(\tau-u-z) \times \mathrm{Str}[X(X+\mathbf{1})\mathbf{P_1}] \exp\left(-\frac{\epsilon}{16}\mathrm{Str}[(\mathbf{1}+2X)\mathbf{K_1}]^2\right), \tag{44}$$

with $u=(x+y)/2$. In addition, we shall use $v=(x-y)/2$ as another new variable, and replace z by $\tau-u$ everywhere, which is admissible because of the presence of the delta function. The integration domains of the radial variables u,v and z are dictated by the hyperbolic symmetry of the saddle point manifold, i.e., we have noncompact integration domains $0<x,y<\infty$ for the bosonic coordinates x,y and a compact integration domain $0<z<1$ for the fermionic coordinate z. For more details on this point, see VWZ. We still have to integrate over the matrix elements of U_1.

V. INTEGRATION OVER THE GRASSMANN VARIABLES

We recall that $U_1=V_1O_1$. Since O_1 commutes with $\mathbf{P}$ and with $\mathbf{K}$, the O_1 integration is trivial, and we are left with the integration over V_1. The parametrization of V_1 in terms of Grassmannian variables and the calculation of the traces in Eq. (44) is quite involved, and is postponed to the Appendixes. Here we note only the results:

$$\begin{aligned}\mathrm{Str}[X(X+\mathbf{1})\mathbf{P_1}] &= 4v(2u+1)B + 2[2u(u+1) - \tau(2u+1-\tau) + v^2] \\ &\quad + 4[\tau(2u+1-\tau)+v^2](A-2\bar{a}),\end{aligned} \tag{45}$$

and

$$\frac{1}{8}\mathrm{Str}[(\mathbf{1}+2X)\mathbf{K_1}]^2 = \tau(2u+1-\tau) - v^2 + 2(A-D)(\tau^2-v^2), \tag{46}$$

where

$$A = \alpha\alpha^* + \beta\beta^*, \quad B = \alpha\alpha^* - \beta\beta^*, \quad D = \iota(\alpha\beta^* - \beta\alpha^*). \tag{47}$$

and $\bar{a}=\alpha\alpha^*\beta\beta^*$. α, α^*, β, β^* are anticommuting variables. It follows that

$$\begin{aligned} e^{-(\epsilon/16)\mathrm{Str}[(\mathbf{1}+2X)\mathbf{K_1}]^2} &= e^{-(\epsilon/2)[\tau(2u+1-\tau)-v^2+2(A+D)(\tau^2-v^2)]} \\ &= [1-\epsilon(A-D)(\tau^2-v^2)]e^{-(\epsilon/2)[\tau(2u+1-\tau)-v^2]}. \end{aligned} \tag{48}$$

These results are inserted into Eq. (44). The measure is given by

$$d\mu(U_1) = 2\pi\, d\mu(V_1) \propto d\alpha\, d\alpha^*\, d\beta\, d\beta^*. \tag{49}$$

Therefore only the terms proportional to $\bar{a}$ survive the integration over the antisymmetric variables. We obtain

$$\langle f_\epsilon(\tau)\rangle \propto \frac{1}{N}\int \mu(u,v,z)\delta(\tau-u-z)e^{-(\epsilon/2)[\tau(2u+1-\tau)-v^2]} \times [1+\epsilon(\tau^2-v^2)][\tau(2u+1-\tau)+v^2]du\, dv\, dz, \tag{50}$$

which is almost our final result.

VI. RESULT AND DISCUSSION

The final result is obtained by an VWZ-like integral [see VWZ, Eq. (8.10)] and is given in the present case by

$$\begin{aligned}\langle f_\epsilon(\tau)\rangle &= 2\int_{\mathrm{Max}(0,\tau-1)}^{\tau} du \int_0^u \frac{v\, dv}{\sqrt{[u^2-v^2][(u+1)^2-v^2]}} \\ &\quad \times \frac{(\tau-u)(1-\tau+u)}{(v^2-\tau^2)^2}[1+\epsilon(\tau^2-v^2)] \\ &\quad \times [\tau(2u+1-\tau)+v^2]e^{-(\epsilon/2)[\tau(2u+1-\tau)-v^2]}. \end{aligned} \tag{51}$$

The constant of proportionality was fixed by the condition $f_\epsilon(0)=1$ (see Ref. [9]). The only difference of Eq. (51) to Ref. [9], where a GOE perturbation was considered, is the additional factor $[1+\epsilon(\tau^2-v^2)]$ in the integrand, and a minus sign with the v^2 term in the exponent, where in the GOE case there is a plus sign.

Figure 1 shows the fidelity decay for different perturbations, as calculated from Eq. (51), together with the result from the exponentiated linear response approximation. For comparison the fidelity decay for the case of a GOE perturbation [8] is shown as well. We see that the linear response approximation is able to describe the fidelity decay for quite a long time very well. For still larger times the linear response approximation underestimates the decay, as compared to the exact result, but still the decay is by orders of magnitude slower as for a GOE perturbation.

Figure 2 shows the fidelity $\langle f_\epsilon(\tau)\rangle$ for three fixed values 0.5, 1.0, 1.5 of τ as a function of the perturbation ϵ. The figure demonstrates that the freezing effect not unexpectedly becomes less and less pronounced with increasing perturbation, although the decay is always by orders of magnitude slower than for the case of a GOE perturbation (not shown). It is further seen that the linear response approximation works very well up to about half the Heisenberg time, but underestimates the decay more and more for increasing τ values.

We thus can conclude that the fidelity freeze is not an artifact of the linear response approximation but is also present in the exact calculation. Due to the perfect and well established correspondence between random matrices and chaotic quantum systems [15–17], this result provides an important new mechanism of preserving quantum stability.

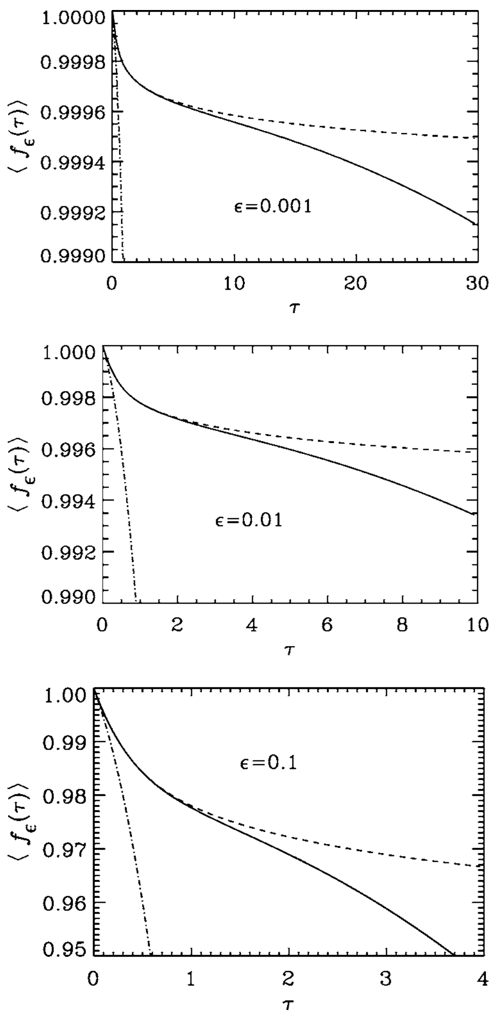

FIG. 1. Ensemble average of the fidelity amplitude $\langle f_\epsilon(\tau)\rangle$ with H_0 taken from the GOE and a purely imaginary antisymmetric perturbation [solid line, calculated from Eq. (51)] for different perturbation strengths ϵ. For comparison, the result from the linear response approximation (dashed line), and for a GOE perturbation (dashed-dotted line) are shown as well.

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APPENDIX A: CALCULATION OF $\mathrm{Str}[X(X+1)\mathbf{P_1}]$

In terms of Pauli matrices X may be expressed as

$$X = -z\mathbf{1} + \begin{pmatrix} \hat{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{A1}$$

where

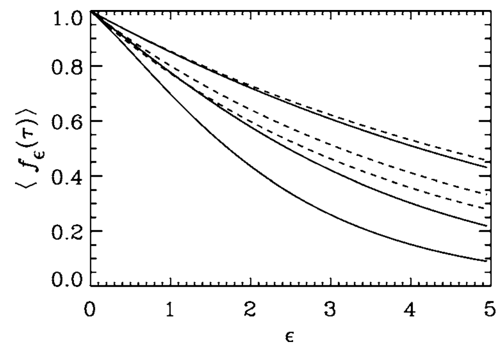

FIG. 2. Ensemble average of the fidelity amplitude $\langle f_\epsilon(\tau)\rangle$ for an imaginary antisymmetric perturbation as a function of ϵ for three fixed values of τ=0.5, 1.0, 1.5 (from top to bottom, solid lines). Again the results from the linear response approximation are shown for comparison (dashed lines).

$$\hat{X} = \begin{pmatrix} x+z & 0 \\ 0 & y+z \end{pmatrix} = \tau\mathbf{1} + v\sigma_z. \tag{A2}$$

It follows that

$$\mathrm{Str}[X(X+\mathbf{1})\mathbf{P_1}] = 4z(z-1) + (1-2z)\mathrm{tr}[\hat{X}(\mathbf{P_1})_{\mathrm{u.l.}}] + \mathrm{tr}[\hat{X}^2(\mathbf{P_1})_{\mathrm{u.l.}}], \tag{A3}$$

where it was used that $\mathrm{Str}\,\mathbf{P_1} = \mathrm{Str}\,\mathbf{P} = 4$, and where $(\mathbf{P_1})_{\mathrm{u.l.}}$ denotes the upper left submatrix of $\mathbf{P_1}$. As was already mentioned, matrices U_1 and U_2 entering the calculation of $\mathbf{P_1}$ and $\mathbf{P_2}$ [see Eq. (40)] are parametrized as

$$U_p = V_p O_p \quad (p = 1,2) \tag{A4}$$

where

$$O_p = \begin{pmatrix} \hat{O}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{A5}$$

and $\hat{O}_1$ and $\hat{O}_2$ are 2×2 orthogonal matrices [see VWZ, Eq. (I.13)]. The matrices V_p may be parametrized as [see VWZ, Eq. (K.26)]

$$(V_p)^{\pm 1} = 1 \pm \iota^{p-1} Y_p + \frac{1}{2}\iota^{2(p-1)} Y_p^2 \pm \frac{1}{2}\iota^{3(p-1)} Y_p^3 + \frac{3}{8} Y_p^4, \tag{A6}$$

where matrices Y_1 and Y_2 are given by

$$Y_p = \begin{pmatrix} \mathbf{0} & -\zeta_p^\dagger \\ \zeta_p & \mathbf{0} \end{pmatrix}, \tag{A7}$$

where

$$\zeta_p = \begin{pmatrix} \alpha_p & \beta_p \\ \alpha_p^* & \beta_p^* \end{pmatrix}, \quad \zeta_p^\dagger = \begin{pmatrix} \alpha_p^* & -\alpha_p \\ \beta_p^* & -\beta_p \end{pmatrix} \tag{A8}$$

[see VWZ, Eqs. (K.23) and (K.25)]. Note the convention $(\alpha^*)^* = -\alpha$ for antisymmetric variables. In VWZ, Eq. (I.13), the sequence of the matrices on the right hand side of Eq. (A4) is reversed. Both parametrizations are equivalent and can be transformed into each other by a straightforward transformation of the α_p, β_p variables.

We are now going to calculate $\mathbf{P_1} = U_1 \mathbf{P} U_1^{-1} = V_1 O_1 \mathbf{P} O_1^{-1} V_1^{-1}$. To simplify notations, we shall omit the lower index "1" in the following. The calculation for $\mathbf{P_2}$ proceeds in the very same way. Since $\mathbf{P}=\mathrm{diag}(1,1,-1,-1)$ commutes with O, we are left with

$$\mathbf{P_1} = V\mathbf{P}V^{-1} = V\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} V^{-1}. \qquad \text{(A9)}$$

For the further calculation it is suitable to introduce the quantities

$$A = \alpha\alpha^* + \beta\beta^*, \quad B = \alpha\alpha^* - \beta\beta^*,$$

$$C = \alpha\beta^* + \beta\alpha^*, \quad D = \iota(\alpha\beta^* - \beta\alpha^*). \qquad \text{(A10)}$$

A, B, C obey the relations

$$A^2 = 2\bar{a}, \quad B^2 = -2\bar{a}, \quad C^2 = -2\bar{a}, \quad D^2 = -2\bar{a}, \qquad \text{(A11)}$$

where $\bar{a} = \alpha\alpha^*\beta\beta^*$, and

$$AB = AC = AD = BC = BD = CD = 0. \qquad \text{(A12)}$$

It follows that

$$\zeta^\dagger\zeta = -A\mathbf{1} - B\sigma_z - C\sigma_x, \quad \zeta\zeta^\dagger = A\mathbf{1}. \qquad \text{(A13)}$$

As a direct consequence we have

$$Y^2 = \begin{pmatrix} -\zeta^\dagger\zeta & \mathbf{0} \\ \mathbf{0} & -\zeta\zeta^\dagger \end{pmatrix} = \begin{pmatrix} A\mathbf{1} + B\sigma_z + C\sigma_x & \mathbf{0} \\ \mathbf{0} & -A\mathbf{1} \end{pmatrix},$$

$$Y^3 = -AY. \qquad \text{(A14)}$$

It follows from Eq. (A6) that

$$V^{\pm 1} = 1 + \left(\frac{1}{2} - \frac{3}{8}A\right)Y^2 \pm \left(1 - \frac{A}{2}\right)Y = \begin{pmatrix} w & \mp\omega^* \\ \pm\omega & \bar{w} \end{pmatrix}, \qquad \text{(A15)}$$

where

$$w = \left(1 + \frac{A}{2} - \frac{3}{4}\bar{a}\right) + \frac{B}{2}\sigma_z + \frac{C}{2}\sigma_x,$$

$$\omega = \left(1 - \frac{A}{2}\right)\zeta, \quad \omega^\dagger = \left(1 - \frac{A}{2}\right)\zeta^\dagger,$$

$$\bar{w} = 1 - \frac{A}{2} + \frac{3}{4}\bar{a}. \qquad \text{(A16)}$$

Inserting the results into Eq. (A9), we have

$$\mathbf{P_1} = \begin{pmatrix} (1 - 4\bar{a} + 2A)\mathbf{1} + 2(B\sigma_z + C\sigma_x) & 2(1-A)\zeta^\dagger \\ 2(1-A)\zeta & (-1 - 4\bar{a} + 2A)\mathbf{1} \end{pmatrix}. \qquad \text{(A17)}$$

A corresponding expression is obtained for $\mathbf{P_2}$. In the average over the antisymmetric variables, only the $\bar{a}$ terms survive, i.e., $\langle\mathbf{P_1}\rangle = \langle\mathbf{P_2}\rangle \propto \mathbf{1}$, as was stated previously.

Inserting finally the upper left corner element of $\mathbf{P_1}$ into Eq. (A3), we end up with Eq. (45).

APPENDIX B: CALCULATION OF Str$[(1+2X)\mathbf{K_1}]^2/8$

It is suitable to write

$$\frac{1}{8}\mathrm{Str}[(1+2X)\mathbf{K_1}]^2$$

$$= \frac{1}{16}\mathrm{Str}[(\mathbf{1}+2X),\mathbf{K_1}]^2 + \frac{1}{8}\mathrm{Str}[(\mathbf{1}+2X)^2\mathbf{K_1}^2]$$

$$= \frac{1}{4}\mathrm{Str}[X,\mathbf{K_1}]^2 + \frac{1}{2}\mathrm{Str}[X(X+\mathbf{1})], \qquad \text{(B1)}$$

where $\mathbf{K_1}^2 = \mathbf{K}^2 = \mathbf{1}$ was used. The second term on the right hand side is easily evaluated:

$$\frac{1}{2}\mathrm{Str}[X(X+1)]$$

$$= \frac{1}{2}[x(x+1) + y(y+1) + 2z(1-z)]$$

$$= u(u+1) + v^2 + z(1-z) = -\tau^2 + (2u+1)\tau + v^2. \qquad \text{(B2)}$$

For the first term on the right hand side we need an expression for $\mathbf{K_1}$. Using $\mathbf{K_1} = U_1\mathbf{K}U_1^{-1}$ [see Eq. (40)] and $U = VO$ [see Eq. (A4)] we may write

$$\mathbf{K_1} = V_1 O_1 \mathbf{K} O_1^{-1} V_1^{-1} = V\mathbf{K}V^{-1}, \qquad \text{(B3)}$$

since K [see Eq. (21)] commutes with O. Using Eq. (A15), we obtain

$$\mathbf{K}_1 = \begin{pmatrix} k & \kappa^\dagger \\ \kappa & \bar{k} \end{pmatrix}, \qquad \text{(B4)}$$

where

$$k = -w\sigma_y w + \omega^\dagger\sigma_z\omega,$$

$$\kappa^\dagger = -w\sigma_y\omega^\dagger - \omega^\dagger\sigma_z\bar{w}, \quad \kappa = -\omega\sigma_y w - \bar{w}\sigma_z\omega,$$

$$\bar{k} = -\omega\sigma_y\omega^\dagger + \bar{w}\sigma_z\bar{w}. \qquad \text{(B5)}$$

Since $\mathbf{K_1}^2 = \mathbf{1}$, we have

$$\frac{1}{4}\mathrm{Str}[X,\mathbf{K}_1]^2 = \frac{1}{2}[\mathrm{Str}(X\mathbf{K}_1)^2 - \mathrm{Str}\,X^2] = \frac{1}{2}[\mathrm{Str}(\hat{X}k)^2 - \mathrm{Str}\,\hat{X}^2], \qquad \text{(B6)}$$

where in the second step expression (A1) for X was used. Using Eqs. (A16) and (B5), we have

$$k = -(1 + A - D)\sigma_y, \qquad \text{(B7)}$$

where $\zeta^\dagger\sigma_z\zeta = D\sigma_y$ was used. Now the calculation of the terms entering the right hand side of Eq. (B6) is straightforward:

$$\frac{1}{2}\mathrm{Str}(\hat{X}k)^2 = (1+2A-2D)(\tau^2 - v^2),$$

$$\frac{1}{2}\mathrm{Str}\,\hat{X}^2 = \tau^2 + v^2. \qquad \text{(B8)}$$

Collecting the results of this section, we have

$$\frac{1}{8}\mathrm{Str}[(\mathbf{1}+2X)\mathbf{K}_1]^2 = \tau(2u+1-\tau) - v^2 + 2(A-D)(\tau^2 - v^2), \qquad \text{(B9)}$$

which follows Eq. (48).

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